Name OF College – S. S. College, J. Bad.

Dept – Mathematics

Topic – Taylor's series, Maclaurin's theorem and problem on it

Time – 9 A.M to 10.30

Date – 30-04-2021

By – Dr. Shree Nath Sharma (Associate prof.)

State and Prove Taylor's series in the infinite form.

Statement :→ If $f(x)$ has finite derivatives of every order in the $(x-\delta, x+\delta)$ containing the point x and

$$R_n = \frac{h^n}{\lfloor n} f^n(x+\theta h) \to 0 \text{ as } n \to \infty \text{ when } |h| < \delta$$

then $f(x+h) = f(x) + \frac{h}{\lfloor 1} f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \dots$

$$\dots + \frac{h^n}{\lfloor n} f^n(x) + \dots \infty$$

Proof :→ By Taylor's theorem with remainder in Lagrange's form

$$f(x+h) = f(x) + \frac{h}{\lfloor 1} f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \dots$$

$$+ \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(x) + \frac{h^n}{\lfloor n} f^n(x+\theta h)$$

$\Rightarrow f(x+h) = S_n + R_n$ where S_n = sum of first n terms

$\lim_{n\to\infty} f(x+h) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} R_n$, $R_n = \frac{h^n}{\lfloor n} f^n(x+\theta h)$

or $f(x+h) = \lim_{n\to\infty} S_n$ $\quad \because R_n \to 0$ as $n \to \infty$

$$= f(x) + \frac{h}{\lfloor 1} f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \dots + \frac{h^n}{\lfloor n} f^n(x) + \dots$$

Statement and Proof of Maclaurin's theorem

Statement :→ If $f(x)$ has finite derivatives of every order in open interval $(-\delta, \delta)$ and $R_n = \frac{h^n}{\lfloor n} f^n(\theta x) \to 0$ as $n \to \infty$ when $|x| < \delta$

then $f(x) = f(0) + \frac{x}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \dots + \frac{x^n}{\lfloor n} f^n(0) + \dots$

Putting $x=0$ and Replacing x for h.
in Taylor's theorem we get the
Maclaurin's theor. infinite series.
and it is nothing but particular
case of Taylor's theorem

failure of Taylor's series

The Expansion of $f(x)$ by Taylor's
theorem will fail for those values of x
for which

(i) $f(x)$ or any its differential
co-efficient becomes infinite

(ii) $f(x)$ or any of its differential
co-efficients is discontinuous.

and (iii) $\lim_{n\to\infty} R_n \neq 0$

i.e $\lim_{n\to\infty} \frac{h^n}{\lfloor n} f^n(a+\theta h) \neq 0$

i.e the series $\sum_{n=0}^{\infty} \frac{h^n}{\lfloor n} f^n(x)$ is not convergent

Failure of Maclaurin's Series :→

The Expansion of $f(x)$ by Maclaurin's
theorem is not valid for those values
of x for which.

(i) $f(0)$ or any of $f'(0), f''(0) \ldots f^n(0)$
is not finite

(ii) $f(x)$ or any of its differential
co-efficients is discontinuous as
x passes through zero.

(iii) $\lim_{n\to\infty} R_n \neq 0$ i.e $\sum_{0}^{\infty} \frac{x^n}{\lfloor n} f^n(0)$ is not convergent

Expansion of Elementary Function

(1) $f(x) = e^x$

$\Rightarrow f'(x) = e^x \quad f''(x) = e^x \quad f^n(x) = e^x$

$f^n(0) = e^0 = 1$

From Maclaurin's theorem with Lagrange's Form of Remainder.

$$f(x) = f(0) + \frac{x}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0)$$

$$+ \frac{x^n}{\lfloor n} f^n(\theta_n x)$$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{\lfloor 2} + \cdots + \frac{x^{n-1}}{\lfloor n-1} + \frac{x^n}{\lfloor n} e^{\theta_n x}$$

Hence the Remainder

$$R_n = \frac{x^n}{\lfloor n} e^{\theta_n x}$$

Now whatever x may be

$$\frac{x^n}{\lfloor n} \to 0 \text{ as } n \to \infty$$

and so Maclaurin's series can be continued to infinity

thus for all x

$$e^x = 1 + x + x^2/\lfloor 2 + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^{n-1}}{\lfloor n-1} + \frac{x^n}{\lfloor n} + \cdots$$

(ii) $f(x) = \cos x$

$f'(x) = -\sin x = \cos(\pi/2 + x)$

$f''(x) = -\sin(\pi/2 + x) = \cos\left(2 \cdot \frac{\pi}{2} + x\right)$

$f^n(x) = \cos[n\pi/2 + x]$

$\therefore f^n(0) = \cos\left(\frac{n\pi}{2}\right) = 0$

$= \pm 1$ according as n is odd or even

of me

If we take $n = 2m$:

Then $f^{2m}(\theta_n x) = \cos\left(\theta_n x + \frac{1}{2} 2m\pi\right) = \cos\left[m\pi + \theta_n x\right]$

$= (-1)^m \cos(\theta_n x)$

according as m is even or odd

From Maclaurin's theorem with Lagrange's Form of Remainder

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0)$$

$$+ \frac{x^n}{\lfloor n} f^n(\theta_n x).$$

$$\cos x = \cos 0 + \frac{x^2}{\lfloor 2}(-\cos 0) + \frac{x^4}{\lfloor 4}(\cos 0) + \cdots$$

$$+ \frac{x^{2m-2}}{\lfloor 2m-2}(-1)^{m-1} + R_n.$$

where $R_n = \frac{x^{2m}}{\lfloor 2m}(-1)^m \cos(\theta_n x)$

$$= 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots \frac{(-1)^{m-1} x^{2m-2}}{\lfloor 2m-2} + R_n$$

(iii) $f(x) = \sin x$

$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots$

$f(x) = \sin x \quad f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x$

$f^{iv}(x) = \sin x$

$\Rightarrow f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = -1$

$$\therefore f(x) = 0 + x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots$$